Automatic Markov Chain Coupling

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Coupling, Theoretically

Besides Gambling, many Probabilists have been interested in Reproduction.
- Grimmett and Stirzaker (1992), 2nd Edition

- Two Markov chains X_t and X'_t are said to have coupled *successfully* if there exists a random time $T < \infty$ such that $X_{T+s} = X'_{T+s}$ for all $s \geq 0$.
- Successful coupling depends on the *implementation*, the actual construction of the two chains X_t and X'_t. It does not follow automatically from the transition probabilities.

Stochastic Recursive Sequences and True Love

- Markov chains X_t with state space E are often built by devising an IID sequence af random maps $F_t : E \to E$ and composing them:

$$X_{t+1} = F_{t+1}(X_t) = F_{t+1}(F_t(X_{t-1})) = \cdots$$

- We then have

$$\mathbb{P}\big(X_{t+1} \in dy \;|X_t = x\big) = \mathbb{P}\big(F_{t+1}(x) \in dy\big). \qquad (1)$$

- Two chains $X_t = F_t \circ F_{t-1} \circ \cdots \circ F_1(x)$ and $X'_t = F_t \circ F_{t-1} \circ \cdots \circ F_1(x')$ built with the same maps F_t may or may not meet. If they do, they become one forever.

Some maps do it, some don't

- For a prescribed set of transition probabilities, there often is an obvious and simple way of generating some IID maps F_t such that (1) holds.

- These maps need not allow the successful coupling of X_t and X_t'. For example, the random walk defined by

$$F_t(x) = x + W_t, \quad \text{where } (W_t) \text{ is i.i.d.}$$

  doesn't allow coupling to occur.

- Instead of guessing a good map F_t, we could start with a bad one and "teach" it to couple!

Couplin' the Field

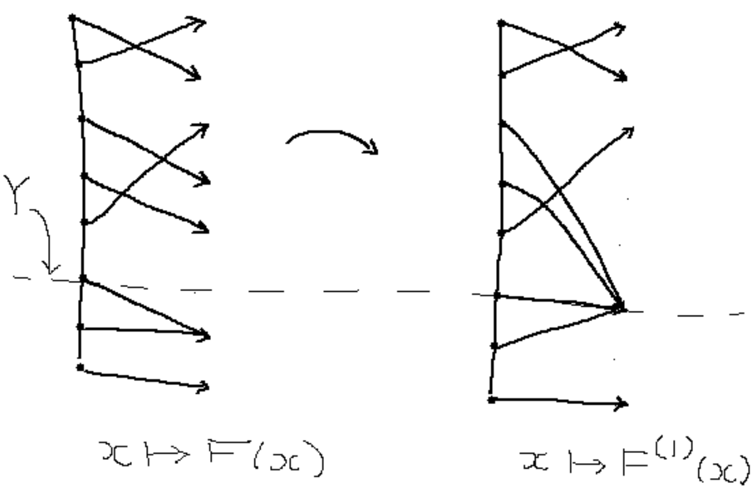
- Start with a random field (map) $F(x)$ whose probabilities are

$$\mathbb{P}\big(F(x) = y\big) = p(x,y).$$

- Take an independent proposal Y with $\mathbb{P}(Y = y) = q(y)$.
- Define

$$F^{(1)}(x) = \begin{cases} Y & \text{if } \frac{p(x,Y)q\left(F(x)\right)}{p\left(x,F(x)\right)q(Y)} > U[0,1], \\ F(x) & \text{otherwise.} \end{cases}$$

- Then $\mathbb{P}\big(F^{(1)}(x) = y\big) = p(x,y)$ also!
- *handcrafted diagram*



More Couplin' the Fields

- To increase the "coupling propensity", we can try iterating this procedure

$$F \to F^{(1)} \to F^{(2)} \to F^{(3)} \to \cdots$$

- To couple Markov chains we teach the individual maps to couple:

$$\begin{array}{ccccc} F_t & \longrightarrow & F_{t+1} & \longrightarrow & F_{t+2} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ F_t^{(1)} & \longrightarrow & F_{t+1}^{(1)} & \longrightarrow & F_{t+2}^{(1)} \end{array}$$

The "taught" chain (downstairs) has the same transition probabilities as the original (upstairs) chain.

Where/When/How/Why does it work?

- So finally, do

$$X_t^{(1)} = F_t^{(1)} \circ F_{t-1}^{(1)} \circ \cdots \circ F_1^{(1)}(x)$$

and

$$X_t'^{(1)} = F_t^{(1)} \circ F_{t-1}^{(1)} \circ \cdots \circ F_1^{(1)}(x')$$

get to couple successfully?

- We'd love to tell you, but unfortunately, this has been

- *Disclaimer.* There are other interesting coupling methods out there, but we don't advertize the competition.