ON A PARABOLIC HARNACK INEQUALITY FOR MARKOV CHAINS

L.A. BREYER

ABSTRACT. For continuous time Markov chains on a countable state space, we derive a parabolic Harnack inequality using probabilistic methods. We derive some consequences of this inequality for the compactness of parabolic (i.e. spacetime harmonic) functions of the process.

1. INTRODUCTION

The parabolic Harnack inequality (see definition in Section 3) is a fundamental tool for the study of the Heat Equation (see Doob, 1984),

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x), \quad t > 0, x \in \mathbb{R}^d. \tag{1}$$

Besides providing bounds on the fundamental solution of (1), it is also the basis for the compactness of the set of solutions $u(t,x)$, as we vary the initial conditions $u(0,\cdot)$, of (1). These functions are known as parabolic functions. In this paper, we prove an analogous inequality for Markov chains, i.e. for the case when the Laplacian in (1) is replaced by the q-matrix of some Markov chain on a countable state space.

From the perspective of probability theory, it is natural to associate with the Laplacian a Brownian motion process X. The parabolic, and more generally superparabolic functions (where the $=$ sign in (1) is replaced by $\geq$) then have an interpretation as excessive functions (see definition in Section 2) for the heat process $(t_0 - t, X_t)$.

A systematic program exists relating the potential theory of the equation (1) with the theory of Brownian motion (e.g. Doob, 1984). Analogously, the Laplacian

Date: January, 1998.

1991 *Mathematics Subject Classification.* Primary 60J, Secondary 60J45.

Key words and phrases. Parabolic Harnack inequality, Markov processes, parabolic functions, positive eigenfunctions.

can be replaced by the generator of some other Markov process X and a potential theory can be developed also (e.g. Dellacherie and Meyer, 1987).

Unfortunately, the generality achieved by the latter theory appears to have left a gap in the case of the parabolic Harnack inequality. This standard inequality is well publicized in the case when X is a diffusion process, and in this case it goes back at least to the work of Moser (1964). When combined with the Ascoli-Arzela theorem, it implies that any set (u_n) of parabolic functions has a convergent subsequence, uniformly on compact sets, to some parabolic function u (Harnack's convergence theorem; see Doob, 1984, 1.XV.11 Theorem (a) in the case of Brownian motion).

A similar result is known for more general Markov processes X, but for sets of superparabolic functions, with convergence to a limiting superparabolic function. There does not appear to be stated explicitly in the probabilistic literature a result dealing with arbitrary sets of *parabolic* functions, even for Markov chains. It is sometimes assumed that the u_n are monotone (increasing or decreasing in n), in which case the limit u clearly exists and is parabolic. We do not make this assumption in Theorem 10.

When X is a Markov chain, this compactness result for parabolic functions has a close connection to the Strong Ratio Limit property or SRLP (see Breyer, 1997, 1998), which is often of interest when Markov chains are applied to the Sciences. The connections between the SRLP and the limiting properties of *quasistationary distributions* are well known (see Pollett, 1993). The SRLP may be viewed geometrically as a property of the Martin boundary of the spacetime process $(t_0 - t, X_t)$, and we shall apply the results of this paper towards this end in another paper (Breyer, 1998).

The plan of the present paper is as follows: Section 2 lists the assumptions, and characterizes parabolic (i.e. spacetime harmonic) functions in terms of the local martingale generator $\mathfrak{A}$. Since parabolic functions are sometimes unbounded, we could not have used a more standard operator, such as the Hille-Yosida infinitesimal generator, whose domain includes only bounded functions.

In Section 3, we prove a parabolic Harnack inequality when X is a Markov chain on a countable state space. We also show that this inequality cannot hold for more general Markov chains.

Section 4 contains a theorem on the compactness of parabolic functions, similar to Harnack's theorem.

2. Markov processes and the local martingale generator

Let $(X_t : t \geq 0)$ denote a Strong Markov process with right continuous sample paths and state space E. We assume that E is a locally compact metric space, and that the transition function $(P_t : t \geq 0)$ of X maps Borel functions into Borel functions. The lifetime of X is

$$\zeta = \inf\{t > 0 : X_t \notin E \text{ or } X_{t-} \notin E\}, \tag{2}$$

where we shall assume that $X_{\zeta+t} \notin E$ for all $t \geq 0$, and we write $\mathbb{P}_\nu$ for the law of X when started with $X_0 \sim \nu$. Given a Borel set $A \subseteq E$, we shall denote by T_A the first hitting time of A, namely $T_A = \inf\{t > 0 : X_t \in A\}$.

A Borel function $f : E \to [0, \infty]$ is called *excessive* if $P_t f \leq f$ and $\lim_{t\downarrow 0} P_t f = f$. We shall assume that X is a *right process* (see Dellacherie and Meyer, 1992). This means that whenever f is excessive, the process $f(X_t)$ *is* is a right continuous $(\mathbb{P}_\nu, \mathcal{F}_t)$-supermartingale. Here $(\mathcal{F}_t : t \geq 0)$ is the usual *completed* filtration, that is the smallest right continuous filtration containing $\mathcal{F}^0_t = \sigma(X_s : s \leq t)$ and such that $\mathcal{F}_0$ contains all ν-null sets, as ν ranges over all possible probability measures on E.

We now define the local martingale generator $\mathfrak{A}$ for the minimal process with (possibly finite) lifetime ζ. It acts on locally bounded Borel functions only.

Definition 1. *A localy bounded Borel function f is said to belong to the domain $\mathcal{D}(\mathfrak{A})$ of the local martingale generator $\mathfrak{A}$ if there exists a Borel function $g(x) =: \mathfrak{A}f(x)$ such that the process*

$$M^f_t = f(X_t)1_{(\zeta>t)} - f(X_0) - \int_0^{t\wedge\zeta} \mathfrak{A}f(X_s)ds$$

is, for each $(\Omega, (\mathcal{F}_t), \mathbb{P}_x)$, a right continuous local martingale up to ζ in the following sense: there exists a sequence of stopping times $T_n \uparrow \zeta$ such that $M^f_{t\wedge T_n}$ is a $(\mathbb{P}_x, \mathcal{F}_t)$ martingale for each $x \in E$.

The "operator" $\mathfrak{A}$ is generally multivalued, since the function $g = \mathfrak{A}f$ can be arbitrary on sets which are visited by the process for a time set of zero Lebesgue measure. At the end of this section, we shall give some examples of well known generators of this type.

We need the following technical lemma for the proof of Theorem 4. In plain language, it signifies that the process cannot leave the state space by a jump when the lifetime is predictable.

Lemma 2. *Let $\tau_n \uparrow \zeta$ be a sequence of $(\mathcal{F}_t)$ stopping times. If $\zeta < \infty$ a.s. , then for every $x \in E$ and compact set $K \subset E$,*

$$\mathbb{P}_x(T_{K^c} = \zeta > \tau_n \ \forall n) = 0.$$

Proof. Consider the predictable stopping time defined by

$$R = \begin{cases} \zeta & \text{on } \{\zeta > \tau_n \ \forall n\}, \\ \infty & \text{otherwise.} \end{cases}$$

Consider the event $A = \{X_{R-} \in E, R < \infty\}$. By (Rogers and Williams, 1994, III.41, Theorem (41.3)), we have

$$1 = (P_0 1_E)(X_{R-}) 1_{(R<\infty)} = \mathbb{E}_x(1_E(X_R), R < \infty | \mathcal{F}_{R-}) \quad a.s. \quad \text{on } A,$$

which is absurd, since the right side of this equation is zero as $X_\zeta \notin E$ on $\{R < \infty\}$. It follows that $X_{\zeta-} \notin E$ on $\{\zeta > \tau_n \ \forall n\}$, and hence on the event $\{T_{K^c} = \zeta > \tau_n \ \forall n\}$, we get the contradiction

$$1 = 1_K(X_{T_{K^c}-}) \leq 1_E(X_{\zeta-}) = 0.$$

This proves the lemma. □

Recall the

Definition 3. *An excessive function f is called harmonic if for every compact set $K \subset E$,*

$$\mathbb{E}_x(f(X_{T_{K^c}}), \zeta > T_{K^c}) = h(x).$$

Theorem 4. *A locally bounded Borel function $f \geq 0$ is harmonic if and only if it belongs to the domain $\mathcal{D}(\mathfrak{A})$ of $\mathfrak{A}$ and satisfies*

$$\mathfrak{A} f = 0 \text{ in } E.$$

Proof. If f is harmonic, it is excessive and the process $f(X_t)1_{(\zeta>t)}$ is a right continuous supermartingale. Because the function is harmonic, we also have by optional stopping

$$f(x) = \mathbb{E}_x(f(X_T), \zeta > T) \leq \mathbb{E}_x(f(X_{t\wedge T}), \zeta > t \wedge T) \leq f(x),$$

whenever T is the first exit time of a compact set. The process $f(X_{t\wedge T})1_{(\zeta>t\wedge T)}$ is thus a martingale. Taking a sequence of compact sets K_n increasing to E and defining $\mathfrak{A}f \equiv 0$, we find that $M^f_{t\wedge T_{K_n^c}}$ is a martingale, so that f satisfies Definition 1 with $T_n = T_{K_n^c} \uparrow \zeta$. Conversely, suppose that $f \in \mathcal{D}(\mathfrak{A})$ and $\mathfrak{A}f = 0$ in E. Let T_n be a localizing sequence as in Definition 1. Take a compact set K and put $T = T_{K^c}$. By the martingale stopping theorem,

$$\begin{aligned} f(x) &= \mathbb{E}_x(f(X_{T\wedge T_n}), \zeta > T \wedge T_n) \\ &= \mathbb{E}_x(f(X_T), \zeta > T, T \leq T_n) + \mathbb{E}_x(f(X_{T_n}), \zeta > T_n, T > T_n). \end{aligned}$$

On $\{T > T_n\}$, the random variable $f(X_{T_n})$ is bounded, since f is locally bounded, so

$$\begin{aligned} \overline{\lim}_{n\to\infty} \mathbb{E}_x(f(X_{T_n}), \zeta > T_n, \sigma > T_n) &\leq \overline{\lim}_{n\to\infty} \|f\|_K \cdot \mathbb{E}_x(\zeta > T_n, \sigma > T_n) \\ &= \|f\|_K \, \mathbb{P}_x(\sigma_K = \zeta > T_n \ \forall n), \end{aligned}$$

which equals zero by Lemma 2 above. Now it suffices to apply the monotone convergence theorem (since f is positive) to get

$$f(x) = \mathbb{E}_x(f(X_T), \zeta > T).$$

It remains only to check that f is excessive. By Fatou's lemma and the assumed right continuity of M^f,

$$\underline{\lim}_{t\downarrow 0} P_t f(x) = \underline{\lim}_{t\downarrow 0} \mathbb{E}_x(f(X_t), \zeta > t) \geq \mathbb{E}_x(f(X_0), \zeta > 0) = f(x),$$

and since $f(X_t)1_{(\zeta>t)}$ is a positive local martingale, it is also a supermartingale, which means $P_t f \leq f$. □

Example 5. Let X_t denote the standard Brownian motion on $\mathbb{R}^d$. By Ito's formula, we have $\mathcal{D}(\mathfrak{A}) \supseteq C^2(\mathbb{R}^d)$ and $\mathfrak{A} = \frac{1}{2}\Delta$ on $C^2(\mathbb{R}^d)$.

Example 6. Let X be a Markov chain on $E = \{1, 2, 3, \ldots\}$ with stable q-matrix (q_{ij}), that is $q_{ii} > -\infty$. Our assumptions on the lifetime imply that X is the *minimal* process with q-matrix q. The generator $\mathfrak{A}$ coincides with the q-matrix (see Rogers and Williams, 1987):

$$\mathfrak{A}f(i) = \sum_{i=1}^{\infty} q_{ij} f(j), \quad i \in E,$$

and its domain $\mathcal{D}(\mathfrak{A})$ includes all functions $f : E \to \mathbb{R}$ such that $\mathfrak{A}f$ is finite. The sequence $T_n \uparrow \zeta$ can be taken as $T_n = T_{K_n^c}$, where the K_n are finite sets such that $K_n \uparrow E$.

Example 7. If X is a Markov process, the associated (backward) spacetime process $\overline{X}$ on $\overline{E} = \mathbb{R} \times E$ is constructed as follows. Let $\overline{\nu}$ be an initial distribution on $\overline{E}$ and choose $(T_0, X_0) \sim \overline{\nu}$. Then set $\overline{X}_t = (T_0 - t, X_t)$. We can similarly define $\overline{X}$ on any open subset of the form $(a, b) \times E$. The spacetime process is again a right process (Sharpe, 1988), with lifetime $\overline{\zeta} = (T_0 - a)^+ \wedge \zeta$, and it is easily shown that the local martingale generator $\overline{\mathfrak{A}}$ of $\overline{X}$ has domain including all functions $f : \overline{E} \to [0, \infty]$ such that $f(\cdot, x)$ is differentiable on $\{f < \infty\}$ and $f(t, \cdot) \in \mathcal{D}(\mathfrak{A})$, and then

$$\overline{\mathfrak{A}}f(t, x) = \mathfrak{A}f(t, x) - \frac{\partial}{\partial t} f(t, x).$$

Consequently, Theorem 4 applies to the process $\overline{X}$. In analogy with (1), a locally bounded Borel function $f : \overline{E} \to [0, \infty]$ such that $f \in \mathcal{D}(\mathfrak{A})$ and

$$\frac{\partial}{\partial t} f(t, x) = \mathfrak{A}f(t, x) \text{ in } (a, b) \times E \tag{3}$$

is called parabolic in (a, b).

3. Parabolic Harnack inequality for chains

In this section, we show that Markov chains on countable state spaces satisfy the following inequality:

Parabolic Harnack inequality: For any compact sets $K \subseteq E$, $\overline{K} \subset (0, \infty) \times E$, let $s > \sup\{t : (t, x) \in \overline{K}\}$. Then there exists a constant $C = C(K, \overline{K})$ such that every function u that is parabolic in $(0, \infty)$ satisfies

$$\sup_{(t,x) \in \overline{K}} u(t, x) \le C \cdot \inf_{y \in K} u(s, y).$$

Let X be an irreducible Markov chain with a countable state space. Here, a set is compact if and only if it is finite. Let K be such a set. Since the state space E is taken to be irreducible, for any $x, y \in K$, there always exists a finite chain of states $x_1, \dots, x_n$ such that X can jump from x to x_1, from x_1 to x_2, ... , $x_n \to y$. We construct a new set $\widetilde{K}$ from K by adding all these states to K, for any combination of states $x, y \in K$. The set $\widetilde{K}$ need not be uniquely determined, but it can always be taken finite, and thus compact, since there are only finitely many ordered pairs (x, y) with $x, y \in K$.

Theorem 8. *Let X be an irreducible Markov chain on a countable state space E, then the parabolic Harnack inequality holds.*

Proof. Let $\delta_0 = s - \sup\{t : \exists x\ (t,x) \in \widetilde{K}\} > 0$, and let $r \geq \delta_0$, $x \in K$ and y such that $(s-r, y) \in \overline{K}$. Without loss of generality, we will assume that $\widetilde{K} \supseteq \{y : \exists t\ (t, y) \in \overline{K}\}$. We denote by $\overline{T}$ the first exit time by $\overline{X}$ from the compact set $[s-r, s] \times \widetilde{K}$. Note that, on the event $\{\overline{T} > r\}$, we have $\overline{T} = \overline{T} \circ \overline{\theta}_r + r$ and $\overline{X}_{\overline{T}} = \overline{X}_{\overline{T}} \circ \overline{\theta}_r$. Since u is parabolic, it follows from Definition 1 and the Markov property that

$$\begin{aligned}
u(s,x) &= \overline{\mathbb{E}}_{(s,x)}(u(\overline{X}_{\overline{T}}), \overline{\zeta} > \overline{T}) \\
&\geq \overline{\mathbb{E}}_{(s,x)}\Big(u(\overline{X}_{\overline{T}}), \overline{\zeta} > \overline{T} > r, \overline{X}_r = (s-r, y)\Big) \\
&= \overline{\mathbb{E}}_{(s,x)}\Big(\overline{\mathbb{E}}_{(s,x)}\big[u(\overline{X}_{\overline{T}} \circ \overline{\theta}_r), \overline{\zeta} \circ \overline{\theta}_r > \overline{T} \circ \overline{\theta}_r \,|\, \overline{\mathcal{F}}_r\big], \overline{T} > r, \overline{X}_r = (s-r, y)\Big) \\
&= \overline{\mathbb{E}}_{(s,x)}\Big(\overline{\mathbb{E}}_{\overline{X}_r}\big[u(\overline{X}_{\overline{T}}), \overline{\zeta} > \overline{T}\big], \overline{T} > r, \overline{X}_r = (s-r, y)\Big) \\
&= \overline{\mathbb{E}}_{(s,x)}(\overline{T} > r, X_r = y)\overline{\mathbb{P}}_{(s-r,y)}(u(\overline{X}_{\overline{T}}), \overline{\zeta} > \overline{T}) \\
&= \mathbb{P}_x(X_r = y, T_{\widetilde{K}^c} > r)u(s-r, y),
\end{aligned}$$

where the probability on the right is nonzero due to the definition of $\widetilde{K}$. Letting $t = s - r$ range over the set $G = \{t : \exists x, (t,x) \in \overline{K}\}$, we get the parabolic Harnack inequality with constant C given by

$$C = \left(\inf_{t \in G, x \in K} \mathbb{P}_x(X_t = y, T_{\widetilde{K}^c} > s - t)\right)^{-1} < \infty.$$

Since u is arbitrary, this concludes the proof. □

The parabolic Harnack inequality cannot hold for all chains on uncountable state spaces, for the following reason: let $\lambda > 0$, and consider the parabolic functions

$$u(t,x) = \mathbb{E}_x(f(X_{t+t_0}), \zeta > t + t_0) = \int P_{t+t_0}(x,dy)f(y),$$

as f ranges over all positive, bounded Borel functions and $t_0 > 0$. Choosing $x, y \in E$ and $\delta_0 > 0$, we find that the resolvent (U_λ) satisfies

$$\begin{aligned} U_\lambda(x,f) &:= \int_0^\infty e^{-\lambda t} P_t(x,f)dt \\ &\le C \int_0^\infty e^{\lambda t} P_{t+\delta_0}(y,f)dt \\ &= Ce^{\lambda\delta_0} \int_{\delta_0}^\infty e^{-\lambda r} P_r(y,f)dr \\ &\le C' U_\lambda(y,f), \end{aligned}$$

which implies that the resolvent measures $\{U_\lambda(x,\cdot) : x \in E\}$ are equivalent (upon interchanging the roles of x and y), i.e. there exists a *reference measure* m on E such that

$$U_\lambda(x,dy) = u_\lambda(x,y)m(dy). \tag{4}$$

4. Application to sets of parabolic functions

In this section, we show some consequences of the existence of a parabolic Harnack inequality.

Suppose we have a sequence (u_n) of parabolic functions, converging to some function u. It is not necessarily true that u is parabolic; indeed if X is a Markov chain with q-matrix (q_{ij}), the parabolic functions are the positive solutions to the equation

$$u(t,x) = u(b,x) + \int_0^t \sum_{y\in E} q_{xy} u(b-s,y)ds. \tag{5}$$

Then if u_n satisfies (5) for each n, we need something like the dominated convergence theorem to assert that $u = \lim_n u_n$ also satisfies (5). A simple condition which guarantees this is that, for each $x \in E$, the measure $\{y\} \mapsto q_{xy}$ be supported by only a finite number of points y. This means probabilistically that there is at most a finite number of destinations for each jump of X. Note that if we assume that u_n is monotone (increasing or decreasing in n), then we do not need conditions on X.

Bounded Jump Condition: For each compact set $K \subset E$, there exists another compact set $K' \subset E$ such that

$$\mathbb{P}_x(X_{T_{K^c}} \in K', \zeta > T_{K^c}) = 1, \quad x \in K,$$

where $T_{K^c} = \inf\{t > 0 : X_t \notin K\}$ is the first exit time from K.

As remarked above, this assumption holds for Markov chains with bounded jumps, for we can take

$$K' = \{y : q_{xy} \neq 0, x \in K\}.$$

The assumption also holds for diffusion processes, for we can take $K' = K$ on account of the continuity of the sample paths. Finally, it obviously holds for the spacetime process $\overline{X}$ whenever it holds for the corresponding process X. We then have the following

Lemma 9. *Let X satisfy the Bounded Jump Condition. If (u_n) is a sequence of parabolic functions which converges boundedly on compacts to a function u, then u is itself parabolic.*

Proof. Let D denote a compact subset of $\overline{E}$, and put $\overline{T} = \inf\{r > 0 : \overline{X}_r \notin D\}$. Since u_n is parabolic,

$$u_n(t, x) = \overline{\mathbb{E}}_{(t,x)}(u_n(\overline{X}_{\overline{T}}), \overline{\zeta} > \overline{T}).$$

Moreover, by the Bounded Jump Condition, $\overline{X}_{\overline{T}}$ belongs to some compact set D' a.s. $\overline{\mathbb{P}}_{(t,x)}$. Thus $u_n(\overline{X}_{\overline{T}})$ is uniformly bounded a.s. , and using the bounded convergence theorem we can let $n \to \infty$ on both sides of the equation. Thus u is itself parabolic. □

Besides the assumptions of Section 2, we suppose now that there exists a σ-finite reference measure m on E for the transition semigroup:

$$\mathbb{E}_x(f(X_t), \zeta > t) = \int p_t(x, y) f(y) m(dy), \quad f \geq 0. \tag{6}$$

This assumption guarantees the existence of a reference measure for the resolvent of the spacetime process $\overline{X}$ on $\overline{E}$. Indeed, we have the analogue of (4),

$$\overline{\mathbb{E}}_{(s,x)} \int_0^{\overline{\zeta}} e^{-\lambda t} f(\overline{X}_t) dt = \int_{\overline{E}} e^{-\lambda(t-s)} 1_{(-\infty,s]}(t) p_{t-s}(x,y) f(t,y) dt m(dy).$$

Recall that a set of potential zero is a set in which the Markov process spends Lebesgue zero time.

Theorem 10. *Suppose that the parabolic Harnack inequality holds together with (6) and the Bounded Jump Condition. Let (u_n) be any set of parabolic functions in (a,b), $b < \infty$, such that there exists a normalizing measure ν supported on $(b,\infty) \times E$ with*

$$\sup_n \int u_n(t,x) \nu(dt \times dx) = 1. \tag{7}$$

There then exists a subsequence $u_{n(k)}$ and a function u, parabolic in (a,b), such that $u_{n(k)}$ converges to u except on a set of potential zero, and boundedly on compact subsets of $(a,b) \times E$.

Proof. Since the (u_n) are parabolic, they are excessive for $\overline{X}$. Consequently, by (6) (see Dellacherie and Meyer, 1987, Chapter XII, Lemma 94, p.81), there exists an excessive function u and a subsequence $u_{n(k)}$ converging to u except on a set of potential zero. By the parabolic Harnack inequality, if $\overline{K} \subset (a,b) \times E$ is compact, we have

$$\sup_{(t,x)\in\overline{K}} \sup_k u_{n(k)}(t,x) \le C,$$

and consequently the convergence is bounded on compacts. Note that u equals $\hat{u}(x) = \underline{\lim}_{k\to\infty} u_{n(k)}$, except on the set of potential zero. Applying Lemma 9, we see that $\hat{u}$ is the desired parabolic function. □

Note that for Markov chains, the convergence of $u_{n(k)}$ to u can easily be shown to occur everywhere, not merely outside a set of potential zero.

5. Acknowledgement

I thank Phil Pollett and Gareth Roberts for invaluable discussions on the topics of this paper. This research was conducted as part of my Ph.D. work.

References

[1] Breyer, L.A. (1997) *Quasistationarity and Conditioned Markov Processes.* Ph.D. Thesis, The University of Queensland.

[2] Breyer, L.A. (1998) *Quasistationarity and Martin boundaries: Conditioned processes.* Preprint available at `http://www.statslab.cam.ac/~laird/`.

[3] Dellacherie, C. and Meyer, P.A. (1987). *Probabilités et Potentiel D. Théorie du potentiel associée à une résolvante. Théorie des processus de Markov.* Hermann, Paris.

[4] Dellacherie, C. and Meyer, P.A. (1992). *Probabilités et Potentiel E. Processus de Markov (fin). Compléments de calcul stochastique.* Hermann, Paris.

[5] Doob, J.L. (1984). *Classical Potential Theory and Its Probabilistic Counterpart.* Springer-Verlag, New York.

[6] Moser, J. (1964). A Harnack inequality for parabolic differential equations. *Comm. Pure Appl. Math.* **17**.

[7] Pollett, P.K. (1993). Recent advances in the theory and application of quasistationary distributions. *Stochastic Models in Engineering, Technology and Management*, S. Osaki and D.N.P. Murthy (eds), World Scientific, Singapore, 477-486.

[8] Rogers, L.C.G. and Williams, D. (1987). *Diffusions, Markov processes, and Martingales, Vol. II.* Wiley, New York.

[9] Walsh, J. and Winkler, W. (1981). Absolute continuity and the fine topology, in *Seminar on Stochastic Processes 1981*, 151-157. Birkhäuser, Boston.

Statistical Laboratory, Department of Pure Mathematics and Mathematical Statistics, 16, Mill Lane, Cambridge CB2 1SB

E-mail address: `L.Breyer@statslab.cam.ac.uk`