QUASISTATIONARY THEOREMS FOR DIFFUSIONS IN A BOUNDED OPEN SET

L.A. BREYER

Abstract. Let X be the minimal diffusion associated with a uniformly elliptic differential operator L on a bounded subdomain of $\mathbb{R}^d$, with C^2 boundary. Under the only assumption that the coefficients of L be Hölder continuous, we prove all the standard quasistationary limit theorems (cf. Markov chain theory). Moreover, we show that the laws of X, conditioned on explosion occuring after time s, converge in total variation, as s tends to infinity, to the law of a positive recurrent diffusion X^∞, which is related to X by the addition of the drift $a\nabla\log\varphi$, where φ is the ground state of L. Previously, such results were shown only for symmetrically reversible diffusions.

1. Introduction

Let E be a bounded subdomain of $\mathbb{R}^d$, with C^2 boundary, and consider a second order elliptic differential operator L on E, having Hölder continuous coefficients. The existence of a 'minimal' diffusion process X associated with L on E is well known, and we denote its lifetime by $\zeta = \inf\{t > 0 : X_t \notin E\}$.

The present paper is concerned with the behaviour of X, when the death time is far into the future. Under this assumption, which is made precise below, it is shown that the process appears positive recurrent to the observer.

An intuitive explanation for this phenomenon can be given as follows: if the process is not allowed to leave the bounded region E, it is condemned to revisit subsets of E endlessly, thus looking like a positive recurrent process.

Consider now the general problem of conditioning X to have an infinite lifetime. This is trivial when $\mathbb{P}_x(\zeta = \infty) > 0$. But already if X is a Brownian motion and $\zeta = \inf\{t > 0 : \|X_t\| > 1\}$, we have $\mathbb{P}_x(\zeta = \infty) = 0$. One hopes to get around this problem by conditioning on the event $\{\zeta > s\}$ for larger and larger values of s. The question then is whether the law of the process, conditioned on these approximating events, converges to the law of some well defined process, as $s \to \infty$. This is indeed the case (corollary 7), as we show here.

The statements above naturally lead to an asymptotic analysis of the distribution functions $t \mapsto \mathbb{P}_x(\zeta > t)$. This task is much simplified when the diffusion is symmetrically reversible (with respect to some finite measure m on E), since the transition function of X then typically has an expansion

$$(1) \qquad \mathbb{E}_x(f(X_t), \zeta > t) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\varphi_i(x)\int_E f(y)\varphi_i(y)m(dy), \qquad f \in L^2(dm),$$

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where the set (φ_i) forms an orthonormal system of eigenfunctions in $L^2(dm)$. Taking $f = 1$, it is then seen that

$$\mathbb{P}_x(\zeta > t) \sim e^{-\lambda_0 t}\varphi_0(x), \quad x \in E, \quad t \to \infty, \tag{2}$$

on account of the inequalities $0 \leq \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$. If X is not symmetrically reversible, an expansion such as (1) need not exist. Nevertheless, we prove here that (2) still holds (see theorem 4) under no further assumptions on L.

Problems of this type are well known in the Markov chain literature, and are usually studied in the framework of quasistationary distributions. The simplest, and best known case in continuous time is that of a Markov chain on a finite state space (Darroch and Seneta (1967)). One then has a large number of quasistationary limit theorems, all of which we can duplicate in the diffusion case (corollary 5, theorem 4). These results also provide a probabilistic interpretation of the principal eigenvalue and associated left and right eigenfunctions of the operator L, with Dirichlet boundary conditions. The most general result (and perhaps most revealing) is theorem 6, which describes the mixing of the path probabilities as t tends to infinity.

Questions related to the quasistationary behaviour of diffusions have been considered previously by various authors. Pinsky (1985) showed the weak convergence of diffusions in a bounded set, conditioned to have infinite lifetime. Our results differ from his in that we specifically do not require the transition function of the process to have an L^2 convergent eigenfunction expansion such as (1). Moreover, we prove convergence in total variation.

In one dimension, diffusions conditioned not to exit an infinite interval were considered recently by Collet et al. (1995) and under less restrictive conditions by Jacka and Roberts (1996). In these circumstances, the limiting process (which is shown to exist) is not recurrent, but 'runs off to infinity'. It is to be noted however, that every one dimensional diffusion is symmetrically reversible.

Other results in countable state space have been obtained by various authors. See the papers by Jacka and Roberts (1995a, 1995b), Jacka et al. (1996) for continuous time, Schrijner and van Doorn (1996) in discrete time, and Tweedie (1974) in discrete time with continuous state space.

2. Assumptions; Analytical preliminaries

Let E be a bounded subdomain of $\mathbb{R}^d$ with C^2 boundary. We consider on E the second order differential operator L defined by

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x^i x^j}f(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x^i}f(x), \quad f \in C^2(E), \tag{3}$$

where $C^2(E)$ denotes the space of twice continuously differentiable real functions on E. The matrix $a_{ij}(x)$ is uniformly positive definite, that is

$$\sum_{i,j=0}^{d} a_{ij}(x)h_i h_j \geq \gamma \left\|h\right\|^2, \quad h \in \mathbb{R}^d,$$

and the functions $x \mapsto a_{ij}(x)$, $x \mapsto b_i(x)$ are assumed locally Hölder continuous for all $i, j \leq d$.

It is known (Azencott (1974)) that there exists a unique minimal diffusion process $X = (\Omega, \mathcal{F}_t, X_t, \zeta, \mathbb{P}_x : x \in E)$, whose generator extends L on $C^2_K(E)$, the space of compactly supported functions in $C^2(E)$. This process has the strong Feller property: for every bounded Borel function f, the function $x \mapsto \mathbb{E}_x f(X_t)$ is continuous.

Let $E_\Delta = E \cup \{\Delta\}$ be the one point compactification of E. It is well known that X can be realized canonically on the space $\Omega = C([0,\infty), E_\Delta)$ of continuous trajectories in E_Δ, as the coordinate process $X_t(\omega) = \omega(t)$, $\omega \in \Omega$. The filtration is then $\mathcal{F}_t = \sigma(X_s : 0 \le s \le t)$, the lifetime is $\zeta = \inf\{t > 0 : X_t = \Delta\}$, and the law $\mathbb{P}_x$, $x \in E$ is uniquely determined by the requirements

$$\mathbb{E}_x(f(X_t) \,|\, \mathcal{F}_s) = \mathbb{E}_{X_s} f(X_{t-s}) \quad \text{a.s. on } \{s < \zeta\},$$

with f a bounded Borel function, and

$$\mathbb{P}_x(X_0 = x, X_{\zeta+t} = \Delta, t \ge 0) = 1.$$

We will also require the shift operators on Ω, $(\theta_t\omega)(s) = \omega(s+t)$.

Suppose that f is a bounded Borel function on E. It is shown in Azencott (1974) that the function $u(t,x) = \mathbb{E}_x(f(X_t), \zeta > t)$ satisfies the Kolmogorov Backward equation

$$\frac{\partial}{\partial t} u(t,x) = Lu(t,x), \quad t > 0, x \in E. \tag{4}$$

The proofs in the next section make crucial use of the parabolic Harnack inequality, which holds for positive solutions of (4) (Friedman (1964)). We state this result in probabilistic form below (proposition 1), but first, it may be useful to describe an analogous inequality for Markov chains.

Suppose that Y_t is a Markov chain in continuous time with lifetime σ, evolving in a discrete state space S. The natural filtration is denoted $(\mathcal{G}_t)$, and we put $p_{ij}(t) = \mathbb{P}_i(Y_t = j)$. We shall denote by $\mathcal{G}^\sigma$ the σ-algebra generated by random variables of the form $H_t 1_{(\sigma > t)}$ where $H_t \in \mathcal{G}_t$. The Markov chain version of the parabolic Harnack inequality states that, for every finite set $K \subseteq S$ and constant $s > 0$, there exists a constant M_K such that

$$\mathbb{E}_i[H] \le M_K \cdot \mathbb{E}_j[H \circ \theta_s], \quad i,j \in K, \quad H \in \mathcal{G}^\sigma_+. \tag{5}$$

The proof of (5) is straightforward from the inequality

$$\mathbb{E}_i[H \circ \theta_s] = \sum_{j \in S} p_{ij}(s) \mathbb{E}_j[H] \ge p_{ik}(s) \mathbb{E}_k[H], \quad i,k \in S.$$

The analogue of (5) for the diffusion X is given below. The σ-algebra $\mathcal{F}^\zeta$ is generated by random variables $H_t 1_{(\zeta > t)}$ where $H_t \in \mathcal{F}_t$.

Proposition 1. *Fix $D \subset\subset E$, a bounded subdomain of the state space. There exists a constant M_D such that, for all positive $\mathcal{F}^\zeta$ measurable random variables H, and $\eta > 0$ sufficiently close to zero, the following inequality holds*

$$\mathbb{E}_x(H \circ \theta_\eta) \le \exp M_D \left(\frac{|x-y|^2}{s} + \frac{s}{\eta} + 1 \right) \cdot \mathbb{E}_y(H \circ \theta_{\eta+s}), \quad x,y \in D, \quad s \ge 0. \tag{6}$$

Note the appearance of an additional shift θ_η, absent in (5). This may be explained as follows. Take $H = f(X_t) 1_{(\zeta > t)}$ for some Borel function f. This function need not be locally bounded; thus, if we wish to bound it by its expectation, we must first 'smooth' it out by shifting it by η time units along the sample path. In the discrete space S however, such a procedure is not necessary, since all measurable functions are locally bounded (in the discrete topology).

The number η can be chosen arbitrarily small, but not arbitrarily large. See Friedman (1964), p.330, theorem 5'.

Proof. Suppose first that $H = f_0(X_0)f_1(X_{t_1})\cdots f_n(X_{t_n})1_{(\zeta>t_n)}$ for some $0 < t_1 < t_2 < \cdots < t_n$, and bounded positive Borel functions $f_0, \ldots, f_n$. The function $u(t,x) = \mathbb{E}_x(H \circ \theta_{t+\eta})$, with η sufficiently small can be written

$$u(t,x) = \mathbb{E}_x(f_0(X_{t+\eta})\mathbb{E}_{X_{t+\eta}}(f_1(X_{t1})f_2(X_{t_2})\cdots f_n(X_{t_n}), \zeta > t_n), \zeta > t+\eta),$$

and hence satisfies (4) in E. By the parabolic Harnack inequality (Friedman (1964), p.330, theorem 5'), if D is a small ball (with closure strictly contained) in E, there exists a constant M_D for which

$$\log \frac{u(t,x)}{u(t+s,y)} \leq M_D \left(\frac{|x-y|^2}{s} + \frac{s}{\eta} + 1 \right)$$

holds with $s > 0$, and $x, y \in D$. Clearly, the same inequality holds (with different constants) if D is merely bounded with closure strictly contained in E, since one may cover D by a finite number of open balls (the number η may have to shrink a finite number of times). We have therefore shown that (6) holds for H of the specified product form. To prove the result for arbitrary positive $\mathcal{F}^\zeta$ measurable H, we simply apply the monotone class theorem, since both sides of (6) are stable under monotone limits. □

Before we end this section, we must introduce one more analytical result. It is known (Pinsky (1995)) that under the assumptions on L and E, there exists a constant $\lambda \geq 0$ and a pair of strictly positive bounded $C^2(E)$ functions φ, φ^* with the following properties:

$$L\varphi(x) = -\lambda\varphi(x), \quad L^*\varphi^*(x) = -\lambda\varphi^*(x) \quad x \in E$$

where L^* is the formal adjoint of L, and $\varphi(x)$, $\varphi^*(x)$ both tend to zero as $x \to \partial E$. The number λ is known as the principal eigenvalue of L, and φ (resp. φ^*) is called the ground state of L (resp. L^*).

Standard arguments (see Breyer and Roberts (1996)) can be used to show that, actually,

$$\mathbb{E}_x\varphi(X_t) = e^{-\lambda t}\varphi(x), \quad \int \varphi^*(y)\mathbb{E}_y h(X_t)dy = e^{-\lambda t}\int h(y)\varphi^*(y)dy \tag{7}$$

for all $x \in E$, and h such that $\int |h(y)|\,\varphi^*(y)dy < \infty$. Indeed, the diffusion X, while transient, is λ-positive recurrent in a way entirely analogous to the case of Markov chains (see Anderson(1991) for the corresponding theory).

3. Limit theorems

We will need to control the size of various expectations of the diffusion, uniformly in the time variable. This is the purpose of the following lemma.

Lemma 2. *let ν be any nonzero excessive measure ($\nu P_t \leq \nu$) for the diffusion process X_t. If $f \in L^1(d\nu)$, then for every subdomain $D \subset\subset E$ with $\nu(D) > 0$,*

$$\sup_{x\in D} \mathbb{E}_x(|f(X_t+\eta)|\,, \zeta > t+\eta) \leq M_D \int |f|\,d\nu, \quad t \geq 0$$

for $\eta > 0$ arbitrarily small and some constant $M_D = M_D(\eta)$ independent of f or ν.

Note that excessive measures always exist. In particular, the measure $\varphi^*(y)dy$ is one such, by (7). But lemma 2 also applies to any *recurrent* diffusion.

Proof. Suppose first that f is a positive Borel function. By proposition 1, we have

$$\mathbb{E}_x(f(X_{t+\eta}), \zeta > t+\eta) \le C_D \mathbb{E}_y(f(X_{t+\eta+s}), \zeta > t+\eta+s), \quad x, y \in D$$

where $s > 0$ is fixed and C_D is some constant independent of t. Integrating both sides with respect to $\nu(dy)$ on D, we have

$$\begin{aligned}\nu(D)\mathbb{E}_x(f(X_{t+\eta}), \zeta > t+\eta) &\le C_D \int_D \nu(dy)\mathbb{E}_y(f(X_{t+\eta+s}), \zeta > t+\eta+s) \\ &\le C_D \mathbb{E}_\nu(f(X_{t+\eta+s}), \zeta > t+\eta+s) \\ &\le C_D \int f d\nu.\end{aligned}$$

Then the conclusion holds with $M_D = C_D/\nu(D)$ when f is positive, and hence in general. □

Thus we see that, even when f is unbounded (but ν-integrable), the expectations $\mathbb{E}_x f(X_{t+\eta})$ are uniformly bounded in t: at any one fixed time, it is very difficult to 'catch' the process visiting regions with very large f-values. Note that the lemma is false if $\eta = 0$, since it would imply that an arbitrary integrable Borel function f is locally bounded.

As an illustration of the usefulness of lemma 2, we give a simple proof of the following extension to the standard limit theorem in (Pinsky (1995), theorem 4.9.9): note that this is known, and can be derived directly using, for example, the corresponding results in Meyn and Tweedie (1993), by using the skeleton chain $Z_n = X_{\epsilon n}$. We shall use proposition 3 in the proof of theorem 4 below.

Proposition 3. *Suppose that X is positive recurrent on E, with invariant measure ν. Then, for all $f \in L^1(d\nu)$,*

$$\lim_{t\to\infty} \mathbb{E}_x f(X_t) = \int f d\nu, \quad x \in E \tag{8}$$

Proof. Theorem 4.9.9 in Pinsky(1995) states that (8) holds for all bounded Borel functions f. If $f \in L^1(d\nu)$ is arbitrary but positive, take a sequence of bounded Borel functions f_n increasing pointwise to f. We can write, by lemma 2,

$$\mathbb{E}_x f_n(X_t) \le \mathbb{E}_x f(X_t) \le \mathbb{E}_x f_n(X_t) + M \int (f - f_n) d\nu,$$

for some constant M. Taking limits as $t \to \infty$ on all sides, we get

$$\int f_n d\nu \le \lim_{t\to\infty} \mathbb{E}_x f(X_t) \le \int f_n d\nu + M \int (f - f_n) d\nu.$$

It remains only to apply the monotone convergence theorem.

When f is not necessarily positive, we write $f = f^+ - f^-$ in the usual manner, and apply the previous paragraph. □

Suppose now that $\mathbb{P}_x(\zeta = \infty) < 1$ for all $x \in E$. Due to the connectedness of E, the diffusion is irreducible, and the lifetime is either infinite, or finite, simultaneously for all $x \in E$. Thus we have $\mathbb{P}_x(\zeta = \infty) = 0$ for all $x \in E$. It follows easily that

$\mathbb{E}_x(f(X_t), \zeta > t)$ converges to zero as t tends to infinity. The next result describes the precise manner in which the convergence occurs.

Theorem 4. *Suppose that $\mathbb{P}_x(\zeta = \infty) < 1$ for all $x \in E$. Let λ, φ and φ^* be the principal eigenvalue, ground state for L and ground state for L^* respectively. Then*

$$\lim_{t\to\infty} e^{\lambda t}\mathbb{E}_x[f(X_t), \zeta > t] = \varphi(x)\int f(y)\varphi^*(y)dy, \quad x \in E, \tag{9}$$

for all Borel functions f such that $\int f(y)\varphi^*(y)dy < \infty$.

Proof. Consider the transition function defined by

$$Q_t f(x) = \varphi^{-1}(x)e^{\lambda t}\mathbb{E}_x(f(X_t)\varphi(X_t), \zeta > t). \tag{10}$$

Let $\mathbb{Q}_x$ be the laws of the associated Markov process. By Girsanov's theorem, the coordinate process X_t is, under $\mathbb{Q}_x$, a diffusion with generator

$$Af(x) = Lf(x) + \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial}{\partial x^i}\log\varphi(x)\frac{\partial}{\partial x^j}f(x), \quad f \in C_K^2(E). \tag{11}$$

Moreover, as remarked in the previous section, we know that $\mathbb{E}_x\varphi(X_t, \zeta > t) = e^{-\lambda t}\varphi(x)$, from which we deduce that $Q_t 1 = 1$, and hence X is recurrent under $\mathbb{Q}_x$. Also, the measure $\nu(dy) = \varphi(y)\varphi^*(y)dy$ is a finite invariant measure under $\mathbb{Q}_x$, hence X is positive recurrent. We normalize φ^* so that this becomes a probability measure. By proposition 3, for any Borel function f such that $\int |f|\,\varphi^{-1}d\nu < \infty$, we have

$$\begin{aligned}\lim_{t\to\infty} e^{\lambda t}\mathbb{E}_x(f(X_t), \zeta > t) &= \lim_{t\to\infty}\varphi(x)\mathbb{Q}_x(f(X_t)\varphi^{-1}(X_t)) \\ &= \varphi(x)\int f(y)\varphi^{-1}(y)\nu(dy) \\ &= \varphi(x)\int f(y)\varphi^*(y)dy.\end{aligned}$$

Note that the function $f\varphi^{-1}$ is in general unbounded, which explains the need for proposition 3. □

As a consequence, we get the usual (compare with Darroch and Seneta(1967)) quasistationary limit theorems, whose proofs are left to the reader.

Corollary 5. *With notation as in theorem 4,*

1. *For all $x \in E$,*
$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}_x(\zeta > t) = -\lambda;$$
2. *For all $x, y \in E$,*
$$\lim_{t\to\infty}\frac{\mathbb{P}_x(\zeta > t)}{\mathbb{P}_y(\zeta > t)} = \frac{\varphi(x)}{\varphi(y)};$$
3. *For all $x \in E$ and Borel functions f such that $\int f(y)\varphi^*(y)dy < \infty$,*
$$\lim_{t\to\infty}\mathbb{E}_x(f(X_t)\,|\,\zeta > t) = \varphi(x)\int f(y)\varphi^*(y)dy.$$

The following result, whose proof is only a slight extension of that of theorem 4, gives arguably an even clearer picture of the phenomenon.

Theorem 6. *Under the assumptions of theorem 4, let $H \in \mathcal{F}^\zeta$ and $K_s \in \mathcal{F}_s$ be bounded, and write $\mu(dx) = \varphi^*(x)dx$. Then, denoting by $\mathbb{Q}_x$ the laws of the positive recurrent process with semigroup* (10) *and generator* (11)*,*

$$\lim_{t\to\infty} e^{\lambda t}\mathbb{E}_x[K_s \cdot H \circ \theta_t] = \varphi(x)\mathbb{Q}_x[K_s] \cdot \mathbb{E}_\mu[H], \quad x \in E \tag{12}$$

boundedly in compact subsets of E.

Proof. We suppose without loss of generality that K_s and H are positive random variables. Let $\xi(dy) = \mathbb{E}_x(K_s, X_s \in dy, \zeta > t)$, and note that, by applying lemma 2 with the excessive measure μ,

$$\xi(g) = \mathbb{E}_x(K_s \cdot g(X_t), \zeta > t) \le M \cdot \|K_s\|\, \mu(g)$$

for every positive Borel function g. Thus the measure ξ is dominated by a constant multiple $C = M \cdot \|K_s\|$ of μ. We aim to show

$$\lim_{t\to\infty} e^{\lambda t}\mathbb{E}_\xi[H \circ \theta_t] = \xi(\varphi) \cdot \mathbb{E}_\mu[H], \tag{13}$$

which is a restatement of (12). Take a sequence (E_n) of compact subsets such that $E_n \uparrow E$, and write

$$e^{\lambda t}\mathbb{E}_\xi[H \circ \theta_t] = e^{\lambda t}\mathbb{E}_\xi[X_t \in E_n, H \circ \theta_t] + e^{\lambda t}\mathbb{E}_\xi[X_t \notin E_n, H \circ \theta_t]. \tag{14}$$

Since

$$e^{\lambda t}\mathbb{E}_x(X_t \in E_n, H \circ \theta_t) = \varphi(x)\mathbb{Q}_x[\varphi^{-1}(X_t)\mathbb{E}_{X_t}[H], X_t \in E_n],$$

which is a quantity bounded over all of E, the bounded convergence theorem together with (9) in which $f(x) = 1_{E_n}(x)\mathbb{E}_x[H]$ gives

$$\lim_{t\to\infty} e^{\lambda t}\mathbb{E}_\xi[X_t \in E_n, H \circ \theta t] = \xi(\varphi)\mathbb{E}_\mu[X_0 \in E_n, H]. \tag{15}$$

The second term in (14) is bounded, uniformly in t, by

$$Ce^{\lambda t}\mathbb{E}_\mu(X_t \notin E_n, \mathbb{E}_{X_t}[H]) \le C\,\|H\|\,\mu(E\backslash E_n); \tag{16}$$

Combining (15) and (16) together with the monotone convergence theorem as $n \to \infty$ gives (13). □

We have stated (12) for an initial distribution concentrated at x, but the same result also holds for arbitrary compactly supported initial distributions, or more generally, distributions which are bounded by multiples of μ. It suffices to choose ξ in (13) accordingly.

The assertion (12) can be thought of as a generalization of the mixing property enjoyed by positive recurrent diffusions. Indeed, for any such diffusion, the principal eigenvalue (resp. ground state) of the generator L (there are no Dirichlet boundary conditions) is zero (resp. the function $\varphi(x) = 1$). The ground state of the adjoint of L is just the density of the invariant probability. Using these quantities in (12) gives the mixing equation.

As an immediate corollary of theorem 6, we can give the probabilistic interpretation of the process with law $\mathbb{Q}$.

Corollary 7. *Suppose that $\mathbb{P}_x(\zeta < \infty) > 0$ for all $x \in E$. For every bounded, $\mathcal{F}_s$-measurable random variable K_s,*

$$\lim_{t\to\infty} \mathbb{E}_x(K_s \,|\, \zeta > t) = \mathbb{Q}_x(K_s), \quad x \in E \tag{17}$$

where $(\mathbb{Q}_x : x \in E)$ *are the laws of a positive recurrent diffusion* X^∞ *with semigroup and generator given by* (10), (11) *respectively. The invariant measure of* X^∞ *is* $\nu(dy) = \varphi(y)\varphi^*(y)dy$, *and the convergence in* (17) *occurs boundedly on compact subsets of* E.

Proof. Apply theorem 6 with $H = 1_{(\zeta>0)}$. □

Note that, to get the above result from theorem 4, we cannot apply Pinsky's theorem (Pinsky (1985)) directly, as we do not know *a priori* wether the limit (9) occurs *uniformly* on compact subsets $D \subset\subset E$.

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Department of Mathematics, The University of Queensland, QLD 4072, Australia
E-mail address: lab@maths.uq.edu.au